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Closed Form Characteristic Function for General Complex Second-Order Form in Correlated Complex Gaussian Random Variables

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93

PREFACE

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13. ABSTRACT *(Maximum 200 words)*

The properties of the left and right eigenvectors of generalized characteristic value problems with non-Hermitian complex matrices are reviewed, and diagonal expansions of various matrix combinations and inverses are derived. The interrelationships of the solutions of two related non-Hermitian characteristic value problems are determined and found to be simply related to the earlier eigenvectors and their properties.

These results are then applied to the evaluation of the characteristic function of a general complex second-order form in correlated Gaussian random variables. The end result is a compact closed form result involving a finite product and a finite sum, along with one square root and exponential. This expression allows for rapid evaluation of the characteristic function for numerous values of its complex argument.

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TABLE OF CONTENTS

LIST OF SYMBOLS

N	size of vector, (1)
Z (bold)	random column vector, (1)
D_1, D_2	N×1 complex constant column vectors, (1)
$C_1 \cdot\cdot C_4$	N×N complex constant matrices, (1)
c	complex form, (1), (56)
*	conjugate, (1)
T	transpose, (1)
H	Hermitian transpose, (1)
overbar	ensemble average, (2)
α	complex scalar, (2), (59)
$f_c(\alpha)$	characteristic function of c, (2)
M	size of general matrices, (3)
B,C	M×M complex non-Hermitian matrices, (3)
R_m	m-th right complex column eigenvector, (3)
λ_m	m-th complex eigenvalue, (3)
det	determinant, (5)
θ_m	m-th phasor of R_m, (6)
a_m	expansion coefficient, (7)
R	right complex M×M normalized modal matrix, (12)
L_m	m-th left complex column eigenvector, (13)
γ_m	m-th complex eigenvalue, (13)
ϕ_m	m-th phasor of L_m, (17)
L	left complex M×M normalized modal matrix, (18)
Λ	M×M diagonal eigenvalue matrix, (19)
D	M×M complex (diagonal) matrix, (21)

d_m	m-th diagonal element of D, (23), (24)
diag	diagonal matrix, (25)
β	complex scalar, (28)
$\underline{A}$	product C^{-1} B, (43)
$\underline{R},\underline{\Lambda}$	right eigen-solutions corresponding to $\underline{A}$, (43)
$\bar{A}$	product B C^{-1}, (44)
$\bar{L},\bar{\Lambda}$	left eigen-solutions corresponding to $\bar{A}$, (44)
X,Y	N×1 real and imaginary parts of random vector Z, (52)
f_1	first-order form, (52)
H	2N×1 complex constant vector, (53)
W	2N×1 real random vector, (53)
f_2	second-order form, (54)
G	2N×2N complex constant matrix, (55)
G_s	symmetric (complex) part of G, (56)
E	2N×1 real vector, mean of W, (57)
K	2N×2N real covariance matrix of W, (57)
p(W)	probability density function of W, (58)
d	complex determinant, (60)
t	complex auxiliary constant, (60)
V	complex auxiliary 2N×1 vector, (60)
e_m,h_m	m-th complex scalars, (72)
$\chi_c(k)$	k-th cumulant of c, (77)
J	general 2N×2N matrix, (84)
J_r,J_i	real and imaginary parts of J, (84)
Q,γ	eigen-solutions corresponding to J_r, (85)
ξ,ζ	complex scalars in determinant, (92)

CLOSED FORM CHARACTERISTIC FUNCTION FOR GENERAL COMPLEX SECOND-ORDER FORM IN CORRELATED COMPLEX GAUSSIAN RANDOM VARIABLES

INTRODUCTION

The evaluation of the performance of signal processors in Gaussian noise and/or a fading medium frequently centers on finding the ensemble average of an exponential of a complex second-order form; for example, see [1]. Furthermore, this complex form can involve complex vectors with arbitrary means and covariance matrices. Here, we will address and solve the following specific problem.

Random vector Z is $N \times 1$ with an arbitrary mean vector $\overline{Z}$ and covariance matrices $\overline{Z\ Z^H}$ and $\overline{Z\ Z^T}$. Vectors D_1 and D_2 are $N \times 1$ complex constant column vectors, while C_1, C_2, C_3, C_4 are $N \times N$ complex constant matrices. The general complex form c is defined as

$$c = D_1^T\ Z + D_2^T\ Z^* + Z^H\ C_1\ Z + Z^H\ C_2\ Z^* + Z^T\ C_3\ Z + Z^T\ C_4\ Z^* \ . \quad (1)$$

The quantity of interest is the characteristic function $f_c(\alpha)$, defined here as the ensemble average (over Gaussian vector Z),

$$f_c(\alpha) = \overline{\exp(\alpha c)} \ , \quad (2)$$

where α is an arbitrary complex scalar. This characteristic function will be found in its most compact form, convenient for rapid evaluation for numerous values of α.

In order to solve this problem, it is necessary to solve a generalized characteristic value problem, with non-Hermitian complex matrices, for the complex right and left eigenvectors. Accordingly, the next section is devoted to reviewing the properties of generalized characteristic value problems for complex matrices and to developing the necessary expansions of relevant determinants and matrices. These properties and expansions are then used in the succeeding section to derive a closed form result for characteristic function $f_c(\alpha)$ in (2). Some related work appears in [2].

RIGHT AND LEFT EIGENVECTORS OF GENERALIZED CHARACTERISTIC VALUE PROBLEM WITH NON-HERMITIAN COMPLEX MATRICES

Let M×M matrices B and C be complex non-Hermitian, and let matrix C be nonsingular. The right M×1 (column) eigenvectors $\{R_m\}$ and corresponding eigenvalues $\{\lambda_m\}$, both of which are generally complex, of the generalized characteristic value problem satisfy the vector equations

$$B\ R_m = \lambda_m\ C\ R_m \quad \text{for } 1 \le m \le M\ . \tag{3}$$

Since this can be written as

$$(B - \lambda_m\ C)\ R_m = 0 \quad \text{for } 1 \le m \le M\ , \tag{4}$$

the complex eigenvalues $\{\lambda_m\}$ can be found from the determinant

$$\det(B - \lambda\ C) = 0\ . \tag{5}$$

Because matrix C is nonsingular, then det(C) ≠ 0, meaning that the term λ^M appears on the left side of (5). Therefore, there are M roots of (5), denoted by eigenvalues $\{\lambda_m\}$, which are assumed to be distinct.

From (4), it follows that the M elements of the m-th right column eigenvector R_m are proportional to the M cofactors of respective elements of any row of the matrix $B - \lambda_m C$ [3; page 113, problem 71]. Alternatively, eigenvector R_m is proportional to any column of the adjoint matrix of $B - \lambda_m C$. Thus, each R_m is uniquely determined by (4), except for a complex scale factor. The absolute scaling of each R_m is fixed here by requiring it to

have unit length:

$$R_m^H R_m = 1 \quad \text{for } 1 \le m \le M . \tag{6}$$

However, each R_m is still ambiguous within a phasor $\exp(i\theta_m)$, where each θ_m is real but arbitrary.

The set of M right eigenvectors, $\{R_m\}$, are linearly independent when the M eigenvalues $\{\lambda_m\}$ are all distinct. To prove this, assume that the first n eigenvectors are linearly independent, but that the n+1 eigenvector is linearly dependent; that is, assume that we can write

$$R_{n+1} = \sum_{m=1}^{n} a_m R_m , \tag{7}$$

where at least one of the coefficients $\{a_m\}$ is nonzero. Then, pre-multiplying (7) by matrix B, we find

$$B\ R_{n+1} = \sum_{m=1}^{n} a_m B\ R_m , \tag{8}$$

or, using (3),

$$\lambda_{n+1} C\ R_{n+1} = \sum_{m=1}^{n} a_m \lambda_m C\ R_m . \tag{9}$$

Pre-multiply (9) by matrix C^{-1}, since C is nonsingular, to get

$$\lambda_{n+1} R_{n+1} = \sum_{m=1}^{n} a_m \lambda_m R_m . \tag{10}$$

Now, multiply (7) by λ_{n+1}, and subtract it from (10), to obtain

$$\sum_{m=1}^{n} (\lambda_m - \lambda_{n+1})\ a_m R_m = 0 . \tag{11}$$

Because eigenvectors $R_1,\ldots,R_n$ have been assumed linearly independent, all n coefficients in (11) must be zero. But, since at least one $a_m \neq 0$, then at least one of the eigenvalues $\lambda_1,\ldots,\lambda_n$ must equal λ_{n+1}. This contradicts the assumption of distinct eigenvalues. Therefore, assumption (7) is invalid; rather, all eigenvectors are linearly independent. Then, the right M×M normalized modal matrix

$$R \equiv [R_1 \ . \ . \ . \ R_M] \tag{12}$$

is nonsingular. Despite (6), it should be noted that $R^H R \neq I$, in general, due to matrices B and C being non-Hermitian.

The left M×1 (column) eigenvectors $\{L_m\}$ and corresponding eigenvalues $\{\gamma_m\}$, both generally complex, of the generalized characteristic value problem satisfy the vector equations

$$L_m^H B = \gamma_m L_m^H C \quad \text{for } 1 \le m \le M \ , \tag{13}$$

or

$$L_m^H (B - \gamma_m C) = 0 \quad \text{for } 1 \le m \le M \ . \tag{14}$$

The left eigenvalues, $\{\gamma_m\}$, are found from the determinant equation, $\det(B - \gamma\, C) = 0$, which is identical in form to (5). Therefore, the left and right eigenvalues, of the two generalized characteristic value problems (3) and (13) for matrices B and C, are the same; that is,

$$\gamma_m = \lambda_m \quad \text{for } 1 \le m \le M \ . \tag{15}$$

Therefore, (14) can be written as

$$L_m^H (B - \lambda_m C) = 0 \quad \text{for } 1 \le m \le M , \tag{16}$$

meaning that the M elements of left (column) eigenvector L_m are proportional to the M cofactors of respective elements of any row of the matrix $(B - \lambda_m C)^H$. Each L_m is uniquely determined by (16), except for a complex scale factor. The absolute scaling of each L_m is fixed here by requiring it to have unit length:

$$L_m^H L_m = 1 \quad \text{for } 1 \le m \le M . \tag{17}$$

However, each L_m is still ambiguous within a phasor $\exp(i\phi_m)$, where each ϕ_m is real but arbitrary.

The set of M left eigenvectors, $\{L_m\}$, are linearly independent when the M eigenvalues $\{\lambda_m\}$ are all distinct. (The proof parallels that presented for the right eigenvectors in (7) - (12).) Then, the left M×M normalized modal matrix

$$L \equiv [L_1 \; . \; . \; . \; L_M] \tag{18}$$

is nonsingular. Again, despite (17), we should observe that $L^H L \ne I$ in general, due to matrices B and C being non-Hermitian.

PROPERTIES OF EIGENVECTOR MATRICES R AND L

Define M×M complex diagonal eigenvalue matrix

$$\Lambda = \mathrm{diag}[\lambda_1 \; . \; . \; . \; \lambda_M] \; . \tag{19}$$

Then, generalized characteristic value problems (3) and (13) can be written in the compact matrix forms

$$B \; R = C \; R \; \Lambda \; , \qquad L^H \; B = \Lambda \; L^H \; C \; , \tag{20}$$

by means of (12), (15), and (18). All M eigenvalues, $\{\lambda_m\}$, are nonzero if and only if matrix B is nonsingular.

Define complex M×M matrix

$$D = L^H \; C \; R \; . \tag{21}$$

Now, pre-multiply the left equation in (20) by L^H, and post-multiply the right equation in (20) by R. There follows, respectively, upon use of (21),

$$L^H \; B \; R = D \; \Lambda \; , \qquad L^H \; B \; R = \Lambda \; D \; . \tag{22}$$

Since the left-hand sides are identical, we must have $D \; \Lambda = \Lambda \; D$. However, because eigenvalue matrix Λ in (19) is diagonal, with <u>distinct</u> element values, it follows that square matrix $D = [d_{mn}]$ must also be diagonal. The proof of this very important claim is as follows: the m,n element of product $D \; \Lambda$ is $\lambda_n \; d_{mn}$, while the m,n element of $\Lambda \; D$ is $\lambda_m \; d_{mn}$. Because these elements must be equal, this leads to $(\lambda_n - \lambda_m) \; d_{mn} = 0$. But, if all the eigenvalues $\{\lambda_m\}$ are distinct, then we must have $d_{mn} = 0$ for

$m \neq n$; that is, matrix D is diagonal. However, matrix D is generally not a multiple of the identity matrix I.

Denote the m-th diagonal element of D by d_m; that is,

$$D = \mathrm{diag}[d_1 \; . \; . \; . \; d_M] \; . \tag{23}$$

Given the unit-length eigenvectors $\{R_m\}$ and $\{L_m\}$ (obtained, for example, via the procedures mentioned in the sequels to (5) and (16), respectively), we can compute, according to (21) and (23), the complex scalars

$$d_m = L_m^H \, C \, R_m \quad \text{for } 1 \leq m \leq M \; . \tag{24}$$

Then, from (21) - (24), we have the two orthogonality rules

$$L^H \, C \, R = D = \mathrm{diag}[d_1 \; . \; . \; . \; d_M] \; , \tag{25}$$

$$L^H \, B \, R = D \, \Lambda = \mathrm{diag}[d_1\lambda_1 \; . \; . \; . \; d_M\lambda_M] \; . \tag{26}$$

All diagonal elements $\{d_m\}$ of matrix D in (23) - (25) are nonzero. This can be seen by computing the determinant of (25):

$$\det(L^H) \, \det(C) \, \det(R) = \det(D) = \prod_{m=1}^{M} d_m \; . \tag{27}$$

But, since all three matrices on the left side are nonsingular, the left side must be nonzero, meaning that the product on the right side must be nonzero.

Furthermore, by appropriate choices of the arbitrary phasors $\{\theta_m\}$ and $\{\phi_m\}$ associated with the right and left eigenvectors $\{R_m\}$ and $\{L_m\}$, respectively, it is easily arranged for all $\{d_m\}$

calculated via (24) to be real and positive. Then, the diagonal matrix in (25) contains only positive real elements. However, the diagonal matrix in (26) will still generally be complex, because the eigenvalues are complex.

EXPANSIONS OF MATRICES

Various matrix expansions and determinants are possible from the above relations. We start with (20) and find, for complex scalar β, that we can express

$$C - \beta B = C - \beta C R \Lambda R^{-1} = C R (I - \beta \Lambda) R^{-1} . \qquad (28)$$

There immediately follows the determinant

$$\det(C - \beta B) = \det(C) \det(I - \beta \Lambda) = \det(C) \prod_{m=1}^{M} (1 - \beta \lambda_m) . \qquad (29)$$

Once the eigenvalues $\{\lambda_m\}$ have been evaluated from (5), this expression allows for rapid evaluation of the determinant of $C - \beta B$ for *numerous* values of complex scalar β. The eigenvectors are not required for this calculation.

From (21), we have the useful result for the inverse,

$$R^{-1} = D^{-1} L^H C . \qquad (30)$$

When we combine this with (28), we obtain a diagonal expansion for the following matrix:

$$(C - \beta B)^{-1} = R (I - \beta \Lambda)^{-1} R^{-1} C^{-1} = R (I - \beta \Lambda)^{-1} D^{-1} L^H =$$

$$= \sum_{m=1}^{M} \frac{1}{d_m (1 - \beta \lambda_m)} R_m L_m^H ; \quad \beta \neq \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_M} . \qquad (31)$$

That is, terms of the form $R_m L_n^H$, with $m \neq n$, do not appear. Once the right and left eigenvectors have been evaluated, this result allows for rapid evaluation of the inverse of matrix $C - \beta B$ for numerous values of complex scalar β. Special cases

of (31) follow immediately:

$$C^{-1} = \sum_{m=1}^{M} \frac{1}{d_m} R_m L_m^H \, , \qquad B^{-1} = \sum_{m=1}^{M} \frac{1}{d_m \lambda_m} R_m L_m^H \, . \tag{32}$$

The latter expansion is valid if all the eigenvalues $\{\lambda_m\}$ are nonzero, that is, if matrix B is nonsingular.

EXAMPLE

A numerical example of the relationships above, with complex matrix elements, eigenvalues, and eigenvectors is presented below for M = 2. As mentioned above, the phasors associated with the eigenvectors have been chosen to lead to a purely positive real diagonal matrix D. The matrix $L^H R$ has been added to illustrate its nonorthogonal behavior.

$$B = \begin{bmatrix} 30-i10 & i10 \\ -18+i6 & 8-i6 \end{bmatrix} , \qquad C = \begin{bmatrix} 30 & -15+i5 \\ -10 & 9-i3 \end{bmatrix} ,$$

$$\Lambda = \mathrm{diag}[1+i \quad 1-i] \, , \quad R = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & i \\ 1+i & 1+i \end{bmatrix} , \quad L = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & i \\ 2+i & 2+i \end{bmatrix} ,$$

$$R^H R = \begin{bmatrix} 1 & \frac{2+i}{3} \\ \frac{2-i}{3} & 1 \end{bmatrix} , \quad L^H L = \begin{bmatrix} 1 & \frac{5+i}{6} \\ \frac{5-i}{6} & 1 \end{bmatrix} , \quad L^H R = \frac{\sqrt{2}}{6} \begin{bmatrix} 4+i & 3+i2 \\ 3 & 4+i \end{bmatrix} ,$$

$$L^H \, C \, R = D = \frac{10\sqrt{2}}{3} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} ,$$

$$L^H \, B \, R = \Lambda \, D = \frac{10\sqrt{2}}{3} \begin{bmatrix} 1+i & 0 \\ 0 & 2-i2 \end{bmatrix} . \tag{33}$$

APPROXIMATE SOLUTION OF INTEGRAL EQUATION

The results above are relevant and useful if we want to solve the following integral equation for r(y), with nonsymmetric functions b(x,y) and c(x,y):

$$\int dy\ b(x,y)\ r(y) = \lambda \int dy\ c(x,y)\ r(y)\ , \tag{34}$$

We discretize (34), leading to approximation

$$\sum_n b(k\Delta,n\Delta)\ r(n\Delta) = \lambda \sum_n c(k\Delta,n\Delta)\ r(n\Delta)\ . \tag{35}$$

In terms of matrices, this is

$$B\ R = \lambda\ C\ R\ , \tag{36}$$

where column vector $R = [\ .\ .\ .\ r(n\Delta)\ .\ .\ .\ ]^T$ here. This equation will have solutions only for certain values of scalar λ:

$$B\ R_m = \lambda_m\ C\ R_m\ , \quad R_m \equiv [\ .\ .\ .\ r_m(n\Delta)\ .\ .\ .\ ]^T\ . \tag{37}$$

At this point, we have the characteristic value problem posed earlier in (3).

The alternative integral equation for $\ell(x)$, where we integrate on x instead of y, is

$$\int dx\ \ell^*(x)\ b(x,y) = \gamma \int dx\ \ell^*(x)\ c(x,y)\ . \tag{38}$$

This can be discretized, leading to approximation

$$\sum_k \ell^*(k\Delta)\ b(k\Delta,n\Delta) = \gamma \sum_k \ell^*(k\Delta)\ c(k\Delta,n\Delta)\ , \tag{39}$$

or, in terms of matrices,

$$L^H B = \gamma L^H C , \tag{40}$$

where column vector $L = [. . . \lambda(k\Delta) . . .]^T$ here. There are solutions to (40) only for certain values of scalar γ:

$$L_m^H B = \gamma_m L_m^H C , \quad L_m \equiv [. . . \lambda_m(k\Delta) . . .]^T . \tag{41}$$

This problem is now identical to that considered earlier in (13).

ALTERNATIVE FORMS OF GENERALIZED CHARACTERISTIC VALUE PROBLEMS

The generalized characteristic value problems were given in (20) in the compact matrix forms

$$B\,R = C\,R\,\Lambda\,, \qquad L^H\,B = \Lambda\,L^H\,C\,. \tag{42}$$

If we pre-multiply the first equation in (42) by C^{-1}, we obtain an alternative matrix form

$$\underline{A}\,\underline{R} = \underline{R}\,\underline{\Lambda}\,, \qquad \underline{A} \equiv C^{-1}\,B\,, \tag{43}$$

with right eigen-solutions $\underline{R}$ and $\underline{\Lambda}$. The relation of these solutions of (43), to the earlier solutions R and Λ, is taken up in appendix A. The matrix $\underline{A}$ is not necessarily Hermitian, even if both matrices B and C are Hermitian.

Similarly, if we post-multiply the second equation in (42) by C^{-1}, we obtain yet another form,

$$\bar{L}^H\,\bar{A} = \bar{\Lambda}\,\bar{L}^H\,, \qquad \bar{A} \equiv B\,C^{-1}\,, \tag{44}$$

with left eigen-solutions $\bar{L}$ and $\bar{\Lambda}$. The relation of these solutions of (44), to the earlier solutions L and Λ, is also addressed in appendix A. Matrix $\bar{A}$ is not necessarily Hermitian, even if both matrices B and C are Hermitian. For the example given earlier in (33), the matrices defined in (43) and (44) are

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}, \qquad \bar{A} = \frac{1}{5}\begin{bmatrix} 10 & 15+i5 \\ -3+i & 0 \end{bmatrix}. \tag{45}$$

Matrix $\underline{A}$ happens to be real but is nonsymmetric. Its eigenvalues are complex, and are given by Λ in (33).

SPECIAL CASE OF SYMMETRIC (NOT HERMITIAN) MATRICES B AND C

If M×M complex matrices B and C are both symmetric, that is, $B^T = B$ and $C^T = C$, (which is not the Hermitian case), then (20), with the left equation transposed, reads as

$$R^T B = \Lambda R^T C , \quad L^H B = \Lambda L^H C . \tag{46}$$

There follows immediately, for this special case,

$$L^H = R^T , \quad L = R^* , \quad L_m = R_m^* . \tag{47}$$

(Actually, the phasor of each L_m could be different from that of R_m.) These relations allow for simplifications in the numerical calculation of the complex eigenvectors; also, because there is now only one fundamental set R (with $L = R^*$), the scaling of the eigenvectors can be altered to suit our convenience. In particular, we now scale eigenvector matrix R so that diagonal matrix D in (25) becomes the identity matrix. Then, orthogonality relations (25) and (26) reduce to

$$R^T C R = I , \quad R^T B R = \Lambda , \tag{48}$$

while expansion (31) becomes

$$(C - \beta B)^{-1} = \sum_{m=1}^{M} \frac{1}{1 - \beta \lambda_m} R_m R_m^T ; \quad \beta \neq \frac{1}{\lambda_1} , \ldots , \frac{1}{\lambda_M} . \tag{49}$$

EXAMPLES FOR M = 2

The first example has symmetric complex matrices B and C, with the scaling of normalized modal matrix R now taken according to the first relation in (48).

$$B = \frac{1}{2}\begin{bmatrix} -2+i6 & 2 \\ 2 & -1-i3 \end{bmatrix} , \quad C = \frac{1}{2}\begin{bmatrix} i6 & 1-i3 \\ 1-i3 & -1 \end{bmatrix} ,$$

$$\Lambda = \mathrm{diag}[1+i \quad 1-i] , \quad L^{*} = R = \begin{bmatrix} \frac{1}{2} & \frac{i}{\sqrt{2}} \\ \frac{1+i}{2} & \frac{1+i}{\sqrt{2}} \end{bmatrix} ,$$

$$R^{H} R = \begin{bmatrix} \frac{3}{4} & \frac{2+i}{2\sqrt{2}} \\ \frac{2-i}{2\sqrt{2}} & \frac{3}{2} \end{bmatrix} , \quad R^{T} R = \begin{bmatrix} \frac{1+i2}{4} & \frac{i3}{2\sqrt{2}} \\ \frac{i3}{2\sqrt{2}} & \frac{-1+i2}{2} \end{bmatrix} ,$$

$$L^{H} C R = R^{T} C R = I ,$$

$$L^{H} B R = R^{T} B R = \Lambda = \mathrm{diag}[1+i \quad 1-i] . \tag{50}$$

The only orthogonality relations satisfied by the eigenvectors R and L are the latter two, with weighting matrices C and B, respectively. Notice that $R_m^H R_m \neq 1$ any longer; in fact, these values are 3/4, 3/2 for m = 1, 2, respectively.

A second example, where matrix C is now real and symmetric, is presented next; matrix B is complex and symmetric.

$$B = \begin{bmatrix} 1 & 2-i \\ 2-i & 1+i \end{bmatrix} , \quad C = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} ,$$

$$\lambda_1 = .83284 - i.11405 , \quad \lambda_2 = -.63284 + i1.11405 ,$$

$$L^* = R = \begin{bmatrix} -.37548 + i.07156 & .51657 + i.05202 \\ -.38980 - i.09394 & -.67808 + i.05400 \end{bmatrix} ,$$

$$R^H R = \begin{bmatrix} .30687 & .06901 - i.14124 \\ .06901 + i.14124 & .73226 \end{bmatrix} ,$$

$$R^T R = \begin{bmatrix} .27898 + i.01949 & .07171 + i.06008 \\ .07171 + i.06008 & .72102 - i.01949 \end{bmatrix} ,$$

$$R R^T = C^{-1} = \begin{bmatrix} .4 & -.2 \\ -.2 & .6 \end{bmatrix} ,$$

$$R^H C R = \begin{bmatrix} 1.03913 & -i.28249 \\ i.28249 & 1.03913 \end{bmatrix} ,$$

$$R^T C R = I , \quad R^T B R = \Lambda = \mathrm{diag}[\lambda_1 \quad \lambda_2] . \tag{51}$$

CHARACTERISTIC FUNCTION OF MOST GENERAL COMPLEX FORM WITH FIRST-ORDER AND SECOND-ORDER TERMS

Let **Z** be a complex N×1 random vector with real and imaginary parts **X** and **Y**; that is, $Z = X + i\,Y$. Let D_1 and D_2 be complex N×1 constant vectors. Let C_1, C_2, C_3, C_4 be complex N×N constant matrices, which need not be Hermitian or symmetric.

The most general first-order complex form in vector **Z** is

$$f_1 = D_1^T\, Z + D_2^T\, Z^* = (D_1 + D_2)^T\, X + i(D_1 - D_2)^T\, Y = 2\, H^T\, W\,, \quad (52)$$

where

$$H = \frac{1}{2}\begin{bmatrix} D_1 + D_2 \\ i(D_1 - D_2) \end{bmatrix}, \qquad W = \begin{bmatrix} X \\ Y \end{bmatrix}. \quad (53)$$

H is a complex 2N×1 constant vector and is completely arbitrary; that is, every complex element of H can be independently specified. **W** is a <u>real</u> 2N×1 random vector.

The most general second-order complex form in vector **Z** is

$$f_2 = Z^H\, C_1\, Z + Z^H\, C_2\, Z^* + Z^T\, C_3\, Z + Z^T\, C_4\, Z^* = W^T\, G\, W\,, \quad (54)$$

where

$$G = \begin{bmatrix} C_1 + C_2 + C_3 + C_4 & i(C_1 - C_2 + C_3 - C_4) \\ -i(C_1 + C_2 - C_3 - C_4) & C_1 - C_2 - C_3 + C_4 \end{bmatrix}. \quad (55)$$

G is a complex 2N×2N constant matrix, which need not be Hermitian. G is completely arbitrary; that is, every complex element of G can be independently specified. However, as far as

the second-order form in (54) is concerned, only the symmetric part of matrix G is active. That is, we can replace G in (54) by its symmetric part $G_s = (G + G^T)/2$ (which is still complex), without affecting the value of f_2; also, see appendix B. Therefore, we have $f_2 = \mathbf{W}^T G_s \mathbf{W}$.

The general complex form of interest here is

$$c = f_2 + f_1 = \mathbf{W}^T G \mathbf{W} + 2 H^T \mathbf{W} = \mathbf{W}^T G_s \mathbf{W} + 2 H^T \mathbf{W} . \tag{56}$$

The pertinent statistics of real random vector $\mathbf{W}$ are

$$E = \overline{\mathbf{W}} = \begin{bmatrix} \overline{X} \\ \overline{Y} \end{bmatrix} , \quad K = \mathrm{Cov}(\mathbf{W}) = \overline{(\mathbf{W} - \overline{\mathbf{W}})\ (\mathbf{W} - \overline{\mathbf{W}})^T} =$$

$$= \overline{\begin{bmatrix} X - \overline{X} \\ Y - \overline{Y} \end{bmatrix} \begin{bmatrix} X^T - \overline{X}^T & Y^T - \overline{Y}^T \end{bmatrix}} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} . \tag{57}$$

Here, mean E is a real 2N×1 constant vector, while covariance K is a real 2N×2N symmetric constant matrix. Some important aspects about obtaining covariance matrix K from the two M×M complex covariance matrices of complex vector Z are discussed in appendix C.

We assume that Z is a complex Gaussian random vector. Then, X and Y are joint-Gaussian, and 2N×1 real random vector $\mathbf{W}$ is Gaussian with probability density function

$$p(\mathbf{W}) = (2\pi)^{-N} (\det K)^{-\frac{1}{2}} \exp\left(- \tfrac{1}{2}(\mathbf{W} - E)^T K^{-1} (\mathbf{W} - E)\right) . \tag{58}$$

The statistical quantity of interest is, for complex scalar α,

the characteristic function of complex random variable c in (56); in particular, the characteristic function is defined here as the average of the following exponential:

$$f_c(\alpha) \equiv \overline{\exp(\alpha c)} = \overline{\exp\left(\alpha\ W^T\ G_s\ W + \alpha\ 2\ H^T\ W\right)} =$$

$$= \int dW\ p(W)\ \exp\left(\alpha\ W^T\ G_s\ W + \alpha\ 2\ H^T\ W\right) =$$

$$=(2\pi)^{-N}(\det\ K)^{-\frac{1}{2}}\int dW\ \exp\left(-\ \tfrac{1}{2}(W-E)^T K^{-1}(W-E)\ +\ \alpha\ W^T\ G_s\ W + \alpha\ 2\ H^T\ W\right)$$

$$= d^{-\frac{1}{2}}\ \exp(t)\ , \tag{59}$$

where [4; section 8-3]

$$d = \det(I\ -\ 2\ \alpha\ G_s\ K) = \det(I\ -\ 2\ \alpha\ K\ G_s)\ ,$$

$$t = \tfrac{1}{2}\ V^T\ (K^{-1}\ -\ 2\ \alpha\ G_s)^{-1}\ V\ -\ \tfrac{1}{2}\ E^T\ K^{-1}\ E\ ,$$

$$V = K^{-1}\ E + 2\ \alpha\ H\ . \tag{60}$$

Covariance K is a symmetric real 2N×2N matrix, $G_s = (G + G^T)/2$ is a symmetric complex 2N×2N matrix which need not be Hermitian, mean E is a real 2N×1 vector, H and V are complex 2N×1 vectors, and α is a complex scalar.

In general, we must invert 2N×2N real symmetric matrix K. Also, we must invert 2N×2N symmetric complex matrix $K^{-1} - 2\ \alpha\ G_s$, which need not be Hermitian, and which depends on argument α.

If $\overline{X} = 0$, $\overline{Y} = 0$, $D_1 = 0$, $D_2 = 0$, then $E = 0$, $H = 0$, $V = 0$, $t = 0$, and we need only evaluate complex determinant $d = \det(I - 2\ \alpha\ K\ G_s)$, which depends on α.

SPECIAL CASE OF Y = 0

Then, complex forms

$$f_1 = (D_1 + D_2)^T X \equiv 2 D^T X ,$$

$$f_2 = X^T (C_1 + C_2 + C_3 + C_4) X \equiv X^T C X ,$$

$$c = X^T C X + 2 D^T X = X^T C_s X + 2 D^T X . \tag{61}$$

Matrices C and D are complex and completely arbitrary; matrix $C_s = (C + C^T)/2$ is the symmetric part of C.

Identify in the subsection above,

$$2N \to N , \quad W \to X , \quad E \to \overline{X} , \quad G_s \to C_s , \quad H \to D , \quad K \to K_{xx} , \tag{62}$$

thereby getting the characteristic function of c as

$$f_c(\alpha) = \overline{\exp(\alpha c)} = d_x^{-\frac{1}{2}} \exp(t_x) , \tag{63}$$

where

$$d_x = \det(I - 2 \alpha K_{xx} C_s) ,$$

$$t_x = \frac{1}{2} V_x^T \left(K_{xx}^{-1} - 2 \alpha C_s\right)^{-1} V_x - \frac{1}{2} \overline{X}^T K_{xx}^{-1} \overline{X} ,$$

$$V_x = K_{xx}^{-1} \overline{X} + 2 \alpha D . \tag{64}$$

Covariance K_{xx} is a real N×N symmetric matrix, C_s is a symmetric complex N×N matrix which need not be Hermitian, mean $\overline{X}$ is a real N×1 vector, D and V_x are complex N×1 vectors, and α is a complex scalar.

SIMPLIFICATION OF GENERAL CHARACTERISTIC FUNCTION (59) - (60)

In (46), replace integer M by 2N, and replace symmetric 2N×2N matrices B and C by G_s and K^{-1}, respectively. There follows

$$G_s \, R = K^{-1} \, R \, \Lambda \, , \quad L^H \, G_s = \Lambda \, L^H \, K^{-1} \, , \quad L = R^* \, . \tag{65}$$

Given G_s and K, solve this characteristic value problem for 2N×2N complex diagonal eigenvalue matrix Λ and the corresponding 2N×2N eigenvector matrix R. Then, orthogonality relations (48) become

$$R^T \, K^{-1} \, R = I \, , \quad R^T \, G_s \, R = \Lambda \, . \tag{66}$$

Now, in (29), replace β by 2α, in addition to the above. There follows

$$\det(K^{-1} - 2 \, \alpha \, G_s) = \det(K^{-1}) \prod_{m=1}^{2N} (1 - 2 \, \alpha \, \lambda_m) \, . \tag{67}$$

Comparison with determinant d in (60) then yields

$$d = \det(I - 2 \, \alpha \, K \, G_s) = \prod_{m=1}^{2N} (1 - 2 \, \alpha \, \lambda_m) \, . \tag{68}$$

This quantity can be quickly evaluated for many complex values of α, once the 2N eigenvalues in (65) have been determined.

Also, from (49), we obtain the diagonal expansion for the inverse matrix,

$$(K^{-1} - 2 \, \alpha \, G_s)^{-1} = \sum_{m=1}^{2N} \frac{1}{1 - 2 \, \alpha \, \lambda_m} \, R_m \, R_m^T \, . \tag{69}$$

This enables the quantity t in (60) to be expressed as

$$t = \frac{1}{2} \sum_{m=1}^{2N} \frac{1}{1 - 2 \alpha \lambda_m} V^T R_m R_m^T V - \frac{1}{2} E^T K^{-1} E . \tag{70}$$

When we use the definition of vector V in (60), there follows

$$t = \frac{1}{2} \sum_{m=1}^{2N} \frac{(e_m + 2 \alpha h_m)^2}{1 - 2 \alpha \lambda_m} - \frac{1}{2} E^T K^{-1} E , \tag{71}$$

where we have defined complex scalars

$$e_m = E^T K^{-1} R_m , \qquad h_m = H^T R_m , \quad \text{for } 1 \le m \le 2N . \tag{72}$$

For $\alpha = 0$ in (59), we have $f_c(0) = 1$; then, d in (60) or (68) is 1, meaning that t must be zero for $\alpha = 0$. Therefore, the last term in (71) must be expressible as

$$\frac{1}{2} E^T K^{-1} E = \frac{1}{2} \sum_{m=1}^{2N} e_m^2 ; \tag{73}$$

this can also be verified by use of expansion (69) for K (upon setting $\alpha = 0$ there). Combining these results, (71) can be simplified to

$$t = \alpha \sum_{m=1}^{2N} \frac{\lambda_m e_m^2 + 2 e_m h_m + 2 \alpha h_m^2}{1 - 2 \alpha \lambda_m} . \tag{74}$$

This expression can be numerically evaluated very easily for numerous values of complex scalar α, once the scalars in (72) have been determined. Also, (74) is the most compact form for t and indicates the fundamental form for characteristic function $f_c(\alpha)$ in (59).

Combined with (68), the final result for the characteristic function of c is

$$f_c(\alpha) = \left(\prod_{m=1}^{2N} (1 - 2\,\alpha\,\lambda_m)\right)^{-\frac{1}{2}} \exp\left(\alpha \sum_{m=1}^{2N} \frac{\lambda_m\, e_m^2 + 2\, e_m\, h_m + 2\,\alpha\, h_m^2}{1 - 2\,\alpha\,\lambda_m}\right). \tag{75}$$

This is equal to [1; (B-54)] when all the matrices are real. In summary, (75) gives the result for ensemble average, $\overline{\exp(\alpha c)}$, of complex form

$$c = W^T G_s W + 2\, H^T W = D_1^T Z + D_2^T Z^* +$$

$$+ Z^H C_1 Z + Z^H C_2 Z^* + Z^T C_3 Z + Z^T C_4 Z^* , \tag{76}$$

where Z is a random complex vector of size N×1, $\{\lambda_m\}$ and $\{R_m\}$ are the eigenvalues and right eigenvectors, respectively, of 2N×2N generalized characteristic value problem (65), and coefficients $\{e_m\}$ and $\{h_m\}$ are given by (72).

If we expand the logarithm of this characteristic function in (75) in a power series in α, the cumulants $\{\chi_c(k)\}$ of complex form c are found to be

$$\chi_c(1) = \sum_{m=1}^{2N} \left(\lambda_m + \lambda_m\, e_m^2 + 2\, e_m\, h_m\right) ,$$

$$\chi_c(k) = 2^{k-1}\, (k-1)! \sum_{m=1}^{2N} \lambda_m^{k-2}\left[\lambda_m^2 + k\left(\lambda_m\, e_m + h_m\right)^2\right] \quad \text{for } k \ge 2 . \tag{77}$$

This agrees with [1; (B-65)] when all the matrices are real.

In order to use these results in (75) and (77), we must solve (65) for matrices $\Lambda = \mathrm{diag}[\lambda_1 \; . \; . \; . \; \lambda_{2N}]$ and $R = [R_1 \; . \; . \; . \; R_{2N}]$, and then compute scalars $\{e_m\}$ and $\{h_m\}$ by means of (72).

Attempts to derive (75) directly, by converting complex form (56) to a sum of (complex) squares of uncorrelated Gaussian random variables, have been unsuccessful. We have been unable to discover any (complex) linear transformation to accomplish this goal directly. Nevertheless, we have derived closed form result (75), which is the most compact form for the characteristic function of complex random variable c in (56). Additionally, we can obtain the ensemble average

$$\overline{\exp(\beta c^*)} = \overline{\exp(\beta^* c)}^* = [f_c(\beta^*)]^* .$$

EXAMPLE

The exact result in (75) was compared with simulation results for the example given earlier in (51), where M = 2. Specifically, we took N = 1, M = 2N = 2, and

$$G_s = \begin{bmatrix} 1 & 2-i \\ 2-i & 1+i \end{bmatrix}, \quad K = \begin{bmatrix} .4 & -.2 \\ -.2 & .6 \end{bmatrix}, \quad K^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

$$E = \begin{bmatrix} .43 \\ .31 \end{bmatrix}, \quad H = \begin{bmatrix} .77 + i.69 \\ .93 - i.57 \end{bmatrix}. \tag{78}$$

Then, the eigenvalues and eigenvectors corresponding to matrices G_s and K^{-1} are

$$\lambda_1 = .8328409 - i.1140487, \quad \lambda_2 = -.6328409 + i1.1140487,$$

$$L^* = R = \begin{bmatrix} -.3754800 + i.0715611 & .5165670 + i.0520160 \\ -.3897995 - i.0939362 & -.6780828 + i.0539997 \end{bmatrix}. \tag{79}$$

Notice that the mean vector E and the linear transformation vector H are both nonzero for this numerical example. This necessitates the calculation and use of eigenvector matrix R.

For $\alpha = -1.1 + i1.3$, (75) yields the exact result

$$f_c(\alpha) = .241135 - i.008663, \tag{80}$$

while a simulation result based on 1E9 trials of the exponential of (56) gave the estimated value

$$f_c(\alpha) \cong .241208 \pm .000037 - i(.008671 \pm .000028). \tag{81}$$

The ± values are the standard deviations estimated during the simulation.

For the alternative example $\alpha = -2.1 + i3.3$, the exact result is

$$f_c(\alpha) = .746822 - i.410478 \, , \tag{82}$$

while a simulation result based on 1E9 trials of the exponential of (56) gave the estimated value

$$f_c(\alpha) \cong .74619 \pm .00076 - i(.41150 \pm .00076) \; . \tag{83}$$

One point to observe about these estimates are their rather large standard deviations, unless a very large number of independent trials are taken. We will return to this point below, after we have discussed the region of convergence of the multiple integral in (59).

CONVERGENCE OF MULTIPLE INTEGRAL IN (59)

In order that the multiple integral in (59) converge, certain conditions on the complex 2N×2N matrix $K^{-1} - 2\,\alpha\, G_s$ must be satisfied. To determine these conditions, consider general complex 2N×2N matrix J in the following multiple integral:

$$\int dW \exp\left(- W^T J\, W\right) \;, \quad J = J_r + i\, J_i \;. \tag{84}$$

Because only the symmetric part of matrix J is active in (84), J can be taken symmetric <u>without</u> loss of generality. Furthermore, only the <u>real</u> part J_r affects the convergence of (84).

Suppose we solve for the 2N×2N eigenvalue matrix γ and eigenvector matrix Q of the real symmetric matrix J_r:

$$J_r\, Q = Q\, \gamma \;, \quad Q^T Q = I \;, \quad Q^T J_r\, Q = \gamma \;. \tag{85}$$

Then, with $Q = [Q_1 \ldots Q_{2N}]$, we have matrix expansion

$$J_r = Q\, \gamma\, Q^{-1} = Q\, \gamma\, Q^T = \sum_{m=1}^{2N} \gamma_m\, Q_m\, Q_m^T \;, \tag{86}$$

allowing the real part of the exponent in (84) to be developed as

$$W^T J_r\, W = \sum_{m=1}^{2N} \gamma_m\, W^T Q_m\, Q_m^T\, W = \sum_{m=1}^{2N} \gamma_m\, v_m^2 \;, \tag{87}$$

where 2N×1 column vector $V = Q^T W = [v_1 \ldots v_{2N}]^T$. The transformed integral on V will obviously converge only if $\gamma_m > 0$ for all m; that is, all the eigenvalues of real symmetric matrix J_r must be positive. This means that matrix J_r must be positive definite in order for integral (84) to converge.

When we identify J with $K^{-1} - 2\ \alpha\ G_s$ in (59), then

$$J_r = K^{-1} - 2\ \mathrm{Re}(\alpha\ G_s)\ . \qquad (88)$$

For the example in (78), this matrix is equal to

$$J_r = \begin{bmatrix} 3-2\alpha_r & 1-4\alpha_r-2\alpha_i \\ 1-4\alpha_r-2\alpha_i & 2-2\alpha_r+2\alpha_i \end{bmatrix},\quad \alpha = \alpha_r + i\ \alpha_i\ . \qquad (89)$$

This matrix is positive definite only if $\alpha_r < 1.5$ and if

$$5 - 2\ \alpha_r + 10\ \alpha_i - 12\ \alpha_r^2 - 4\ \alpha_i^2 - 20\ \alpha_r\ \alpha_i > 0\ . \qquad (90)$$

These conditions on complex variable α are depicted in figure 1 as the region inside the hyperbola, which is basically in the second quadrant. Any values for α outside this region lead to a divergent integral (59); that is, the random variable $\exp(\alpha c)$ in (59) has an infinite average value when complex parameter α is outside the hyperbola in figure 1. In the one-dimensional case, this is similar to the example of a zero-mean unit-variance Gaussian random variable x with transformed average value

$$\overline{\exp(\alpha x^2)} = \int dx\ (2\pi)^{-\frac{1}{2}} \exp\left(-\ \frac{1}{2}\ x^2 + \alpha\ x^2\right) \qquad (91)$$

being infinite if $\alpha_r > 1/2$.

In general, matrix J_r in (88) must be positive definite in order that (59) converge, that is, in order for random variable $\exp(\alpha c)$ to have a finite average. That latter value is then given by (59) and (60). The region dictated by the positive definiteness of J_r can take on a variety of shapes. For example,

if the 2,2 element, 1+i, of G_s in (78) is replaced by 3-i4, the region of convergence of multiple integral (59) is the interior of a circle of radius 3.25 centered at $\alpha = -1.75 - i2.5$. Of course, these regions always include the origin $\alpha = 0$.

If one tries to estimate the ensemble average of $\exp(\alpha c)$ by an arithmetic average of a large number of independent trials, the standard deviation is found to be rather large, especially near the boundary of convergence of integral (59). In fact, as α approaches the boundary, the standard deviation tends to ∞. This makes the availability of closed form result (75) a very useful tool for analysis, without requiring excessive computer simulation time.

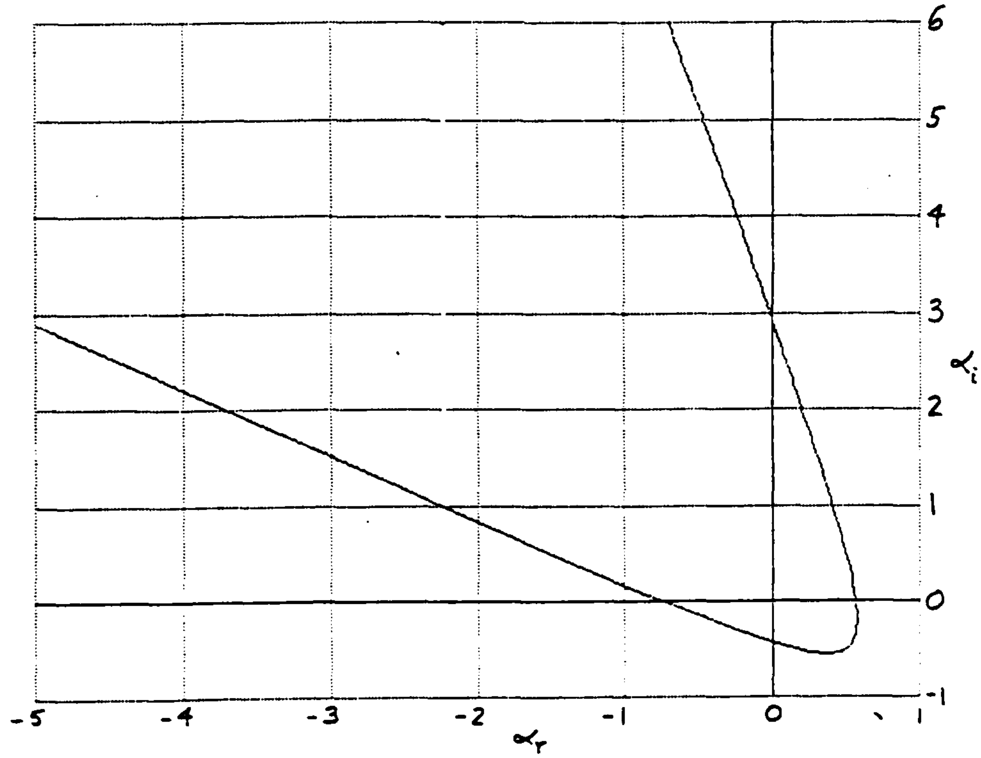

Figure 1. Region of Convergence of (59) for Example (78)

EXTENSION TO JOINT CHARACTERISTIC FUNCTION

The joint characteristic function of the real and imaginary parts, c_r and c_i, of general complex form c in (56) was derived in [1; page 141]. Attempts to simplify that result have not been successful. For example, the determinant in [1; (G-16) - (G-18)] takes the form $\det(I - \xi A - \zeta B)$, where A and B are constant M×M matrices, and ξ and ζ are complex scalars. Guided by (68), we might hope that this determinant would take on the form

$$d = \det(I - \xi A - \zeta B) \overset{?}{=} \prod_{m=1}^{M} \left(1 - \lambda_m \xi - \nu_m \zeta\right) . \tag{92}$$

However, this could never be, because there are not enough degrees of freedom on the right-hand side of (92). To demonstrate this, consider the following example for M = 2:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} , \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} , \tag{93}$$

which are real symmetric matrices. Then, the left side of (92) is given by

$$d = 1 - 5\xi - 7\zeta + 5\xi^2 + 13\xi\zeta + 8\zeta^2 . \tag{94}$$

On the other hand, the right side of (92) is of the form

$$(1 - \alpha\xi - \beta\zeta)(1 - \mu\xi - \nu\zeta) =$$

$$= 1 - (\alpha+\mu)\xi - (\beta+\nu)\zeta + \alpha\mu\xi^2 + (\alpha\nu+\beta\mu)\xi\zeta + \beta\nu\zeta^2 . \tag{95}$$

If we choose constants α,μ,β,ν to match the ξ,ξ^2,ζ,ζ^2 terms, the numerical values for $\alpha\nu+\beta\mu$ must then be $(35\pm\sqrt{85})/2 = 12.89$ or 22.11, which do not equal the desired value of 13 in (94).

SUMMARY

The characteristic function of a general complex second-order form in correlated complex N×1 Gaussian random variables has been evaluated in the compact closed form (75). Its use requires the solution of 2N×2N generalized characteristic value problem (65), with complex non-Hermitian matrices. The final form in (75) can be accomplished by means of a finite product and a finite sum of 2N terms, along with one square root and exponential. Rapid evaluation for numerous values of complex scalar α is readily achieved. The additional scalars required are given in (72).

APPENDIX A. ALTERNATIVE CHARACTERISTIC VALUE PROBLEMS

An alternative form of the eigenvalue problem was derived in (43), namely

$$\underline{A}\ \underline{R} = \underline{R}\ \underline{\Lambda}\ , \quad \text{or} \quad \underline{A}\ \underline{R}_m = \underline{\lambda}_m\ \underline{R}_m \quad \text{for } 1 \le m \le M\ , \tag{A-1}$$

where M×M matrix

$$\underline{A} = C^{-1}\ B. \tag{A-2}$$

Corresponding to these M right eigenvectors $\{\underline{R}_m\}$, there are also the left eigenvectors $\{\underline{L}_m\}$ of matrix $\underline{A}$, which solve the equations

$$\underline{L}_m^H\ \underline{A} = \underline{\gamma}_m\ \underline{L}_m^H \quad \text{for } 1 \le m \le M\ . \tag{A-3}$$

The relationship of these solutions to the earlier results in (3) and (13), for $\{\lambda_m\}$, $\{R_m\}$, and $\{L_m\}$, is the subject of this appendix. However, the presentation will be significantly condensed here, because it parallels, to some extent, the material in the main body of this report.

From (A-1), we find

$$(\underline{A} - \underline{\lambda}_m\ I)\ \underline{R}_m = 0\ , \quad \det(\underline{A} - \underline{\lambda}_m\ I) = 0\ . \tag{A-4}$$

Substituting (A-2), this develops into

$$\det(C^{-1}\ B - \underline{\lambda}_m\ I) = \det(C^{-1})\ \det(B - \underline{\lambda}_m\ C) = 0\ . \tag{A-5}$$

But, since matrix C is nonsingular, reference to (5) yields

$$\underline{\lambda}_m = \lambda_m \quad \text{for } 1 \le m \le M\ , \tag{A-6}$$

under some ordering of the eigenvalues. That is, the two

eigenvalue matrices are equal: $\underline{\Lambda} = \Lambda$.

We can now write (A-1) in the form

$$(\underline{A} - \lambda_m\, I)\, \underline{R}_m = (C^{-1}\, B - \lambda_m\, I)\, \underline{R}_m = C^{-1}\, (B - \lambda_m\, C)\, \underline{R}_m = 0 \quad \text{(A-7)}$$

for $1 \le m \le M$. When we pre-multiply (A-7) by nonsingular matrix C, and compare the result with (4), we see that we must have

$$\underline{R}_m = R_m\, u_m \quad \text{for } 1 \le m \le M\,, \quad \text{or} \quad \underline{R} \equiv [\underline{R}_1\, \ldots\, \underline{R}_M] = R\, U\,, \quad \text{(A-8)}$$

where $U \equiv \text{diag}[u_1\, \ldots\, u_M]$ is an arbitrary diagonal matrix of complex scalar elements. The scaling of $\{\underline{R}_m\}$ may be settled by taking them to be unit length; then, (6) indicates that we would have $|u_m| = 1$ for all m. So, the right eigenvectors $\{\underline{R}_m\}$ in (A-1) can be taken equal to $\{R_m\}$ in (3), if desired.

As for the left eigenvectors $\{\underline{L}_m\}$, develop (A-3) as

$$\underline{L}_m^H\, (\underline{A} - \underline{\gamma}_m\, I) = 0\,, \quad \det(\underline{A} - \underline{\gamma}_m\, I) = 0\,. \quad \text{(A-9)}$$

Comparison with (A-4) - (A-6) yields $\underline{\gamma}_m = \lambda_m$ for $1 \le m \le M$. We can now write (A-9) in the form

$$\underline{L}_m^H\, (\underline{A} - \lambda_m\, I) = \underline{L}_m^H\, (C^{-1}\, B - \lambda_m\, I) = \underline{L}_m^H\, C^{-1}\, (B - \lambda_m\, C) = 0. \quad \text{(A-10)}$$

Comparison with (16) yields the connection

$$\underline{L}_m^H\, C^{-1} = L_m^H\, v_m^*\,, \quad \text{or} \quad \underline{L}_m = C^H\, L_m\, v_m \quad \text{for } 1 \le m \le M\,, \quad \text{(A-11)}$$

using the fact that C is nonsingular, where $\{v_m\}$ are arbitrary complex scalars. In matrix form, (A-11) is

$$\underline{L} = C^H\, L\, V\,, \quad V = \text{diag}[v_1\, \ldots\, v_M]\,. \quad \text{(A-12)}$$

Scalars $\{v_m\}$ in (A-11) can be taken so that left eigenvectors $\{\underline{L}_m\}$ have unit length; that is, $|v_m| = (L_m^H \, C \, C^H \, L_m)^{-\frac{1}{2}}$.

The relation for $\underline{L}$ in (A-12) can be made to look more akin to $\underline{R} = R \, U$ in (A-8), by using (25); namely,

$$\underline{L}^{-H} = C^{-1} \, L^{-H} \, V^{-H} = R \, D^{-1} \, V^{-H} = R \, \tilde{U} \, , \qquad \text{(A-13)}$$

where $\tilde{U} = D^{-1} \, V^{-H}$ is a diagonal matrix.

PROPERTIES OF EIGENVECTOR MATRICES $\underline{R}$ AND $\underline{L}$

The expressions in (A-1) and (A-3) can be put into the forms

$$\underline{A} \, \underline{R} = \underline{R} \, \Lambda \, , \quad \underline{L}^H \, \underline{A} = \Lambda \, \underline{L}^H \, , \qquad \text{(A-14)}$$

with the help of (A-6). Now, define complex M×M matrix

$$\underline{D} = \underline{L}^H \, \underline{R} \, . \qquad \text{(A-15)}$$

Then, by arguments similar to those presented in (20) - (22),

$$\underline{L}^H \, \underline{A} \, \underline{R} = \underline{D} \, \Lambda \, , \quad \underline{L}^H \, \underline{A} \, \underline{R} = \Lambda \, \underline{D} \, , \quad \underline{D} \, \Lambda = \Lambda \, \underline{D} \, , \qquad \text{(A-16)}$$

$$\underline{D} = \mathrm{diag}[\underline{d}_1 \, . \, . \, . \, \underline{d}_M] \, , \qquad \text{(A-17)}$$

$$\underline{d}_m = \underline{L}_m^H \, \underline{R}_m = u_m \, v_m^* \, L_m^H \, C \, R_m = u_m \, v_m^* \, d_m \quad \text{for } 1 \le m \le M \, . \qquad \text{(A-18)}$$

Therefore, the magnitude is known: $|\underline{d}_m| = |u_m \, v_m^* \, d_m|$.

Thus, we have the two orthogonality relations

$$\underline{L}^H \, \underline{R} = \underline{D} \, , \quad \underline{L}^H \, \underline{A} \, \underline{R} = \underline{D} \, \Lambda \, , \qquad \text{(A-19)}$$

where $\underline{D}$ and Λ are diagonal. Elements $\{\underline{d}_m\}$ can be made positive.

EXPANSIONS OF MATRICES

A number of useful expansions and relations can now be developed from the above results. From (A-14) and (A-15),

$$\underline{A}^n = \underline{R}\ \Lambda^n\ \underline{R}^{-1} = \underline{R}\ \Lambda^n\ \underline{D}^{-1}\ \underline{L}^H = \sum_{m=1}^{M} \frac{\lambda_m^n}{\underline{d}_m}\ \underline{R}_m\ \underline{L}_m^H\ . \tag{A-20}$$

Similarly, for complex scalar β,

$$\underline{I} - \beta\ \underline{A}^n = \underline{R}\ (I - \beta\ \Lambda^n)\ \underline{R}^{-1} = \underline{R}\ (I - \beta\ \Lambda^n)\ \underline{D}^{-1}\ \underline{L}^H =$$

$$= \sum_{m=1}^{M} \frac{1 - \beta\ \lambda_m^n}{\underline{d}_m}\ \underline{R}_m\ \underline{L}_m^H\ , \tag{A-21}$$

$$\det(I - \beta\ \underline{A}^n) = \det(I - \beta\ \Lambda^n) = \prod_{m=1}^{M} (1 - \beta\ \lambda_m^n)\ , \tag{A-22}$$

$$(I - \beta\ \underline{A}^n)^{-1} = \underline{R}\ (I - \beta\ \Lambda^n)^{-1}\ \underline{R}^{-1} = \underline{R}\ (I - \beta\ \Lambda^n)^{-1}\ \underline{D}^{-1}\ \underline{L}^H =$$

$$= \sum_{m=1}^{M} \frac{1}{\underline{d}_m\left(1 - \beta\ \lambda_m^n\right)}\ \underline{R}_m\ \underline{L}_m^H\ . \tag{A-23}$$

These diagonal expansions allow rapid calculation for numerous values of complex scalar β, once the eigenvalues $\{\lambda_m\}$ and right and left eigenvectors $\{\underline{R}_m\}$ and $\{\underline{L}_m\}$ have been determined.

EXAMPLE

For the example considered earlier in (33), we have the following results:

$$\underline{A} = C^{-1} B = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} ,$$

$$\Lambda = \mathrm{diag}[1+i \quad 1-i] , \quad \underline{R} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & i \\ 1+i & 1+i \end{bmatrix} , \quad \underline{L} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1-i & -1+i \\ i & 1 \end{bmatrix} ,$$

$$\underline{R}^H \underline{R} = \begin{bmatrix} 1 & \frac{2+i}{3} \\ \frac{2-i}{3} & 1 \end{bmatrix} , \quad \underline{L}^H \underline{L} = \begin{bmatrix} 1 & \frac{-2-i}{3} \\ \frac{-2+i}{3} & 1 \end{bmatrix} ,$$

$$\underline{L}^H \underline{R} = \underline{D} = \frac{2}{3} I , \quad \underline{L}^H \underline{A} \underline{R} = \underline{D} \Lambda = \frac{2}{3} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} . \qquad \text{(A-24)}$$

The matrix $\underline{A}$ happens to be real; however, its eigenvalues are complex. Relations (A-8) and (A-11) have been verified for this example.

PROPERTIES OF ALTERNATIVE PRODUCT $\bar{A} = B\ C^{-1}$

Another form of the eigenvalue problem was stated in (44):

$$\bar{L}^H \bar{A} = \bar{\Lambda}\ \bar{L}^H , \quad \text{or} \quad \bar{L}_m^H \bar{A} = \bar{\lambda}_m \bar{L}_m^H \quad \text{for } 1 \le m \le M , \tag{A-25}$$

where M×M matrix

$$\bar{A} = B\ C^{-1} . \tag{A-26}$$

Corresponding to these left eigenvectors $\{\bar{L}_m\}$, there are also the right eigenvectors $\{\bar{R}_m\}$ of matrix $\bar{A}$, which solve the equations

$$\bar{A}\ \bar{R}_m = \bar{\gamma}_m \bar{R}_m \quad \text{for } 1 \le m \le M . \tag{A-27}$$

This can be developed as follows:

$$(\bar{A} - \bar{\gamma}_m I)\ \bar{R}_m = 0 , \quad \det(\bar{A} - \bar{\gamma}_m I) = 0 , \tag{A-28}$$

$$\det(B\ C^{-1} - \bar{\gamma}_m I) = \det(B - \bar{\gamma}_m C)\ \det(C^{-1}) = 0 . \tag{A-29}$$

Therefore, from (5), $\bar{\gamma}_m = \lambda_m$ for all m. This enables the development of (A-28) according to

$$(\bar{A} - \lambda_m I)\ \bar{R}_m = (B\ C^{-1} - \lambda_m I)\ \bar{R}_m = (B - \lambda_m C)\ C^{-1}\ \bar{R}_m = 0. \tag{A-30}$$

Using (4) and the nonsingular character of matrix C, we find

$$C^{-1}\ \bar{R}_m = R_m\ u_m , \quad \bar{R}_m = C\ R_m\ u_m , \quad \bar{R} = C\ R\ U , \tag{A-31}$$

where $U = \text{diag}[u_1 \ . \ . \ . \ u_M]$. This can be compared with result $\underline{R} = R\ U$ in (A-8).

Regarding the left eigenvectors $\{\bar{L}_m\}$, (A-25) yields

$$\bar{L}_m^H (\bar{A} - \bar{\lambda}_m I) = 0 , \quad \det(\bar{A} - \bar{\lambda}_m I) = 0 . \tag{A-32}$$

Comparison with (A-28) - (A-29) yields $\bar{\lambda}_m = \lambda_m$ for all m. Then, (A-32) yields

$$\bar{L}_m^H (\bar{A} - \lambda_m I) = \bar{L}_m^H (B C^{-1} - \lambda_m I) = \bar{L}_m^H (B - \lambda_m C) C^{-1} = 0. \tag{A-33}$$

When we post-multiply (A-33) by nonsingular matrix C, we obtain

$$\bar{L}_m^H (B - \lambda_m C) = 0 . \tag{A-34}$$

Defining $V = \mathrm{diag}[v_1 \ . \ . \ . \ v_M]$, reference to (16) reveals that

$$\bar{L}_m^H = L_m^H v_m^* , \quad \bar{L}_m = L_m v_m , \quad \bar{L} = L V . \tag{A-35}$$

This can be compared with result $\underline{L} = C^H L V$ in (A-12).

PROPERTIES OF EIGENVECTOR MATRICES $\bar{R}$ AND $\bar{L}$

The expressions in (A-27) and (A-25) can be written as

$$\bar{A}\,\bar{R} = \bar{R}\,\Lambda\,, \quad \bar{L}^H\,\bar{A} = \Lambda\,\bar{L}^H\,. \tag{A-36}$$

Now, define complex M×M matrix

$$\bar{D} = \bar{L}^H\,\bar{R}\,. \tag{A-37}$$

Then, by arguments similar to those presented in (20) - (22),

$$\bar{L}^H\,\bar{A}\,\bar{R} = \bar{D}\,\Lambda\,, \quad \bar{L}^H\,\bar{A}\,\bar{R} = \Lambda\,\bar{D}\,, \quad \bar{D}\,\Lambda = \Lambda\,\bar{D}\,, \tag{A-38}$$

$$\bar{D} = \mathrm{diag}[\bar{d}_1\ .\ .\ .\ \bar{d}_M]\,, \tag{A-39}$$

$$\bar{d}_m = \bar{L}_m^H\,\bar{R}_m = u_m\,v_m^*\,L_m^H\,C\,R_m = u_m\,v_m^*\,d_m \quad \text{for } 1 \le m \le M\,. \tag{A-40}$$

Thus, we have the two orthogonality relations

$$\bar{L}^H\,\bar{R} = \bar{D}\,, \quad \bar{L}^H\,\bar{A}\,\bar{R} = \bar{D}\,\Lambda\,, \tag{A-41}$$

where both $\bar{D}$ and Λ are diagonal. The elements $\{\bar{d}_m\}$ can be arranged to be positive real for all m.

In summary, the desired interrelationships between the alternative characteristic value problems are

$$\underline{R} = R\,U\,, \quad \underline{L} = C^H\,L\,V\,, \quad \text{for } \underline{A} = C^{-1}\,B\,, \tag{A-42}$$

while

$$\bar{R} = C\,R\,U\,, \quad \bar{L} = L\,V\,, \quad \text{for } \bar{A} = B\,C^{-1}\,. \tag{A-43}$$

Conversely, having solved for $\underline{R},\underline{L}$ or $\bar{R},\bar{L}$, these equations can be used to solve for R,L.

EXAMPLE

For the example considered earlier in (33), we have the following results:

$$\bar{A} = B\ C^{-1} = \frac{1}{5}\begin{bmatrix} 10 & 15+i5 \\ -3+i & 0 \end{bmatrix} ,$$

$$\Lambda = \mathrm{diag}[1+i \quad 1-i] \ , \quad \bar{R} = \frac{1}{\sqrt{60}}\begin{bmatrix} 5-i5 & 5-i5 \\ 1+i3 & -3+i \end{bmatrix} , \quad \bar{L} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -i \\ 2+i & -2-i \end{bmatrix} ,$$

$$\bar{R}^H\ \bar{R} = \begin{bmatrix} 1 & \frac{5+i}{6} \\ \frac{5-i}{6} & 1 \end{bmatrix} , \quad \bar{L}^H\ \bar{L} = \begin{bmatrix} 1 & \frac{-5-i}{6} \\ \frac{-5+i}{6} & 1 \end{bmatrix} ,$$

$$\bar{L}^H\ \bar{R} = \bar{D} = \frac{\sqrt{10}}{6}\ I\ , \quad \bar{L}^H\ \bar{A}\ \bar{R} = \bar{D}\ \Lambda = \frac{\sqrt{10}}{6}\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} . \tag{A-44}$$

Relations (A-31) and (A-35) have been verified for this example.

ALTERNATIVE FORM INVOLVING SQUARE ROOT OF C

If we pre-multiply (20) by $C^{-\frac{1}{2}}$, and rearrange the equation, we arrive at

$$C^{-\frac{1}{2}} \, B \, C^{-\frac{1}{2}} \left(C^{\frac{1}{2}} \, R\right) = \left(C^{\frac{1}{2}} \, R\right) \Lambda \, , \qquad \text{(A-45)}$$

which is a more symmetric form. This leads us to consider the matrix $C^{-\frac{1}{2}} \, B \, C^{-\frac{1}{2}}$ and its right and left eigen-solutions:

$$\tilde{A} \equiv C^{-\frac{1}{2}} \, B \, C^{-\frac{1}{2}} \, , \quad \tilde{A} \, \tilde{R} = \tilde{R} \, \tilde{\Lambda} \, , \quad \tilde{L}^H \, \tilde{A} = \tilde{\gamma} \, \tilde{L}^H \, . \qquad \text{(A-46)}$$

By a process similar to that employed above, we find

$$\tilde{\gamma} = \tilde{\Lambda} = \Lambda \, , \quad \tilde{R} = C^{\frac{1}{2}} \, R \, U \, , \quad \tilde{L} = C^{\frac{1}{2}H} \, L \, V \, , \qquad \text{(A-47)}$$

where U and V are arbitrary diagonal matrices. This can be compared with (A-42) and (A-43).

APPENDIX B. GENERAL SECOND-ORDER REAL AND COMPLEX FORMS

Suppose K×1 column vector X and K×K arbitrary matrix R are real. Let

$$R_s = \frac{1}{2}(R + R^T) \;, \quad R_d = \frac{1}{2}(R - R^T) \;. \tag{B-1}$$

Then,

$$R_s^T = R_s \;, \quad R_d^T = - R_d \;. \tag{B-2}$$

Consider the real scalar quantity

$$r = X^T R X = X^T (R_s + R_d) X = r_s + r_d \;, \tag{B-3}$$

where scalars

$$r_s = X^T R_s X \;, \quad r_d = X^T R_d X \;. \tag{B-4}$$

But,

$$r_d = r_d^T = X^T R_d^T X = - X^T R_d X = - r_d \;. \tag{B-5}$$

Therefore, $r_d = 0$, meaning that $r = r_s$, or

$$r = X^T R X = X^T R_s X \;. \tag{B-6}$$

Thus, we need only consider the symmetric part of matrix R, at least as far as quadratic form r is concerned; that is, only the symmetric part of matrix R is active in quadratic form (B-3).

More generally, suppose that K×1 column vector Z and K×K arbitrary matrix C are complex. Let

$$C_s = \frac{1}{2}(C + C^H) \;, \quad C_d = \frac{1}{2}(C - C^H) \;. \tag{B-7}$$

Then,

$$C_s^H = C_s \;, \quad C_d^H = - C_d \;. \tag{B-8}$$

Now, consider the complex scalar quantity

$$c = Z^H C Z = Z^H (C_s + C_d) Z = c_s + c_d , \tag{B-9}$$

where complex scalars

$$c_s = Z^H C_s Z , \quad c_d = Z^H C_d Z . \tag{B-10}$$

But,

$$c_s^* = c_s^H = Z^H C_s^H Z = Z^H C_s Z = c_s , \tag{B-11}$$

meaning that c_s is real; also,

$$c_d^* = c_d^H = Z^H C_d^H Z = - Z^H C_d Z = - c_d , \tag{B-12}$$

meaning that c_d is purely imaginary. Thus, we can express (B-9) as

$$c = Z^H C Z = c_r + i c_i , \tag{B-13}$$

where the two scalars

$$c_r = Z^H C_s Z , \quad c_i = -i Z^H C_d Z , \tag{B-14}$$

are both purely real quantities. (If matrix C is Hermitian, then $C^H = C$, $C_s = C$, $C_d = 0$, and $c_i = 0$, making c a Hermitian form.)

It should be noted that c, above in (B-9), is not the most general complex second-order form; the latter is

$$c = Z^H C_1 Z + Z^H C_2 Z^* + Z^T C_3 Z + Z^T C_4 Z^* , \tag{B-15}$$

where C_1, C_2, C_3, C_4 are arbitrary complex K×K matrices.

APPENDIX C. TWO COMPLEX COVARIANCES OF A STATIONARY COMPLEX RANDOM PROCESS

Let $z(t)$ be a zero-mean stationary complex random process with a complex covariance, defined in the usual way as

$$\overline{z(t)\ z^*(t - \tau)} = \overline{z\left(t + \frac{\tau}{2}\right)\ z^*\left(t - \frac{\tau}{2}\right)} \equiv R_{zz}(\tau) = R_{zz}^*(-\tau) \ . \qquad \text{(C-1)}$$

However, there is insufficient information in complex function $R_{zz}(\tau)$ to determine the real auto and cross covariances of the stationary real and imaginary components of $z(t)$. That is, with $z(t) = x(t) + i\ y(t)$, (C-1) yields the two real equations

$$R_{xx}(\tau) + R_{yy}(\tau) = \text{Re}\ R_{zz}(\tau) \quad \text{(even)}\ ,$$

$$R_{yx}(\tau) - R_{xy}(\tau) = \text{Im}\ R_{zz}(\tau) \quad \text{(odd)}\ . \qquad \text{(C-2)}$$

But, we cannot extract any of the four real covariances on the left side of (C-2), even when we account for the fact that $R_{yx}(\tau) = R_{xy}(-\tau)$. Functions $R_{xx}(\tau)$ and $R_{yy}(\tau)$ are always even functions of τ, but $R_{xy}(\tau)$ need not be odd in τ. (As an example, consider $y(t) = x(t - t_o)$.)

In order to allow for a unique solution of (C-2), we must also know the <u>complementary</u> complex covariance of process $z(t)$,

$$\overline{z(t)\ z(t - \tau)} = \overline{z\left(t + \frac{\tau}{2}\right)\ z\left(t - \frac{\tau}{2}\right)} \equiv \underline{R}_{zz}(\tau) = \underline{R}_{zz}(-\tau) \ . \qquad \text{(C-3)}$$

(For a real process $z(t)$, $R_{zz}(\tau)$ and $\underline{R}_{zz}(\tau)$ are identical.) The complementary information in (C-3) yields the two additional real

equations

$$R_{xx}(\tau) - R_{yy}(\tau) = \text{Re } \underline{R}_{zz}(\tau) \quad \text{(even)} \; ,$$

$$R_{yx}(\tau) + R_{xy}(\tau) = \text{Im } \underline{R}_{zz}(\tau) \quad \text{(even)} \; . \tag{C-4}$$

The solutions to simultaneous equations (C-2) and (C-4) are now available as

$$R_{xx}(\tau) = \frac{1}{2} \text{ Re}\left[R_{zz}(\tau) + \underline{R}_{zz}(\tau)\right] \; ,$$

$$R_{yy}(\tau) = \frac{1}{2} \text{ Re}\left[R_{zz}(\tau) - \underline{R}_{zz}(\tau)\right] \; ,$$

$$R_{yx}(\tau) = \frac{1}{2} \text{ Im}\left[R_{zz}(\tau) + \underline{R}_{zz}(\tau)\right] \; ,$$

$$R_{xy}(\tau) = \frac{1}{2} \text{ Im}\left[\underline{R}_{zz}(\tau) - R_{zz}(\tau)\right] \; . \tag{C-5}$$

In the special case where complementary covariance $\underline{R}_{zz}(\tau)$ is identically zero, (C-5) simplifies to

$$R_{xx}(\tau) = R_{yy}(\tau) = \frac{1}{2} \text{ Re } R_{zz}(\tau) \quad \text{(even)} \; ,$$

$$R_{yx}(\tau) = -R_{xy}(\tau) = \frac{1}{2} \text{ Im } R_{zz}(\tau) \quad \text{(odd)} \; . \tag{C-6}$$

When two zero-mean stationary complex random processes $z(t)$ and $w(t)$ are involved, the situation is slightly more complicated. We now define the two complex cross covariances

$$\overline{z(t)\; w^*(t - \tau)} \equiv R_{zw}(\tau) = R^*_{wz}(-\tau) \; ,$$

$$\overline{z(t)\; w(t - \tau)} \equiv \underline{R}_{zw}(\tau) = \underline{R}_{wz}(-\tau) \; . \tag{C-7}$$

Then, if we express the complex random processes in terms of their real and imaginary parts according to

$$z(t) = x(t) + i\,y(t)\ , \qquad w(t) = u(t) + i\,v(t)\ , \tag{C-8}$$

relation (C-7) develops into

$$R_{xu}(\tau) + R_{yv}(\tau) + i\,R_{yu}(\tau) - i\,R_{xv}(\tau) = R_{zw}(\tau)\ ,$$

$$R_{xu}(\tau) - R_{yv}(\tau) + i\,R_{yu}(\tau) + i\,R_{xv}(\tau) = \underline{R}_{zw}(\tau)\ , \tag{C-9}$$

the solutions of which are

$$R_{xu}(\tau) = \frac{1}{2}\,\mathrm{Re}\left[R_{zw}(\tau) + \underline{R}_{zw}(\tau)\right]\ ,$$

$$R_{yv}(\tau) = \frac{1}{2}\,\mathrm{Re}\left[R_{zw}(\tau) - \underline{R}_{zw}(\tau)\right]\ ,$$

$$R_{yu}(\tau) = \frac{1}{2}\,\mathrm{Im}\left[R_{zw}(\tau) + \underline{R}_{zw}(\tau)\right]\ ,$$

$$R_{xv}(\tau) = \frac{1}{2}\,\mathrm{Im}\left[\underline{R}_{zw}(\tau) - R_{zw}(\tau)\right]\ . \tag{C-10}$$

The four relations in (C-10) must be augmented with those in (C-5), as well as with analogous ones giving the auto and cross covariances of u(t) and v(t) in terms of $R_{ww}(\tau)$ and $\underline{R}_{ww}(\tau)$. The end result is that enough information is available from the <u>six</u> complex covariance functions $R_{zz}(\tau)$, $\underline{R}_{zz}(\tau)$, $R_{ww}(\tau)$, $\underline{R}_{ww}(\tau)$, $R_{zw}(\tau)$, $\underline{R}_{zw}(\tau)$, to evaluate <u>all</u> of the auto and cross covariances between the four zero-mean stationary real processes x(t), y(t), u(t), and v(t).

EXAMPLE

Consider the pair of zero-mean stationary complex processes $\zeta(t)$ and $\omega(t)$ filtered by arbitrary transfer functions $A(f)$ and $B(f)$, respectively, as shown in figure C-1. The corresponding outputs are $z(t)$ and $w(t)$, respectively.

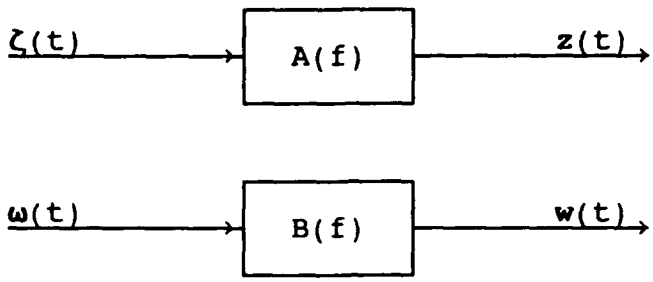

Figure C-1. Two-Channel Filtering Operation

The two input complex cross covariances are

$$R_{\zeta\omega}(\tau) = \overline{\zeta(t)\ \omega^*(t - \tau)}\ , \quad \underline{R}_{\zeta\omega}(\tau) = \overline{\zeta(t)\ \omega(t - \tau)}\ . \qquad \text{(C-11)}$$

The two corresponding input complex cross spectra are

$$G_{\zeta\omega}(f) = \int d\tau\ \exp(-i2\pi f\tau)\ R_{\zeta\omega}(\tau)\ ,$$

$$\underline{G}_{\zeta\omega}(f) = \int d\tau\ \exp(-i2\pi f\tau)\ \underline{R}_{\zeta\omega}(\tau)\ . \qquad \text{(C-12)}$$

It may be shown that the two corresponding output cross spectra for the complex processes in figure C-1 satisfy the rules

$$G_{zw}(f) = G_{\zeta\omega}(f)\ A(f)\ B^*(f)\ ,$$

$$\underline{G}_{zw}(f) = \underline{G}_{\zeta\omega}(f)\ A(f)\ B(-f)\ . \qquad \text{(C-13)}$$

Now, if the product of filter transfer functions $A(f)$ and $B(-f)$ is zero for all f, then the complementary output cross spectrum $\underline{G}_{zw}(f)$ is zero for all f. This leads to complementary output cross covariance $\underline{R}_{zw}(\tau)$ being zero for all τ, as utilized in (C-6). A case where this situation arises is when both transfer functions $A(f)$ and $B(f)$ correspond to one-sided filters, nonzero only for $f > 0$. Processes $z(t)$ and $w(t)$ in figure C-1 are then analytic processes, because auto spectra $G_{zz}(f)$ and $G_{ww}(f)$ and cross spectrum $G_{zw}(f)$ are zero for $f < 0$; that is, auto covariances $R_{zz}(\tau)$ and $R_{ww}(\tau)$ and cross covariance $R_{zw}(\tau)$ are analytic in the upper-half τ-plane.

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